Professor Ford has proved that if the generated area is to be passed over, to the greatest possible extent, but *two* times, AB should be rotated about the intersection of the perpendicular bisectors of AB and $A'B'$.

DISCUSSIONS.

Doctor Morris gives a device for a perpetual calendar, with an account of the theory on which it is constructed. A similar explanation will hold for any of the many forms of such calendars. It is believed that this article will be of interest, since explanations of the theory of perpetual calendars are seldom given.

Professor Bennett gives a set of simple identities from which a four-place logarithm table can readily be calculated. The method is probably new. It bears some relation to the plan used by Briggs (cf. *Encyclopædia Britannica*, 11th ed., vol. 16, pp. 875–876); but Briggs needed first to calculate and use explicitly the modulus, while Professor Bennett does not make any use of the modulus.

I. The Theory of Perpetual Calendars.

By Frank R. Morris, University of California.

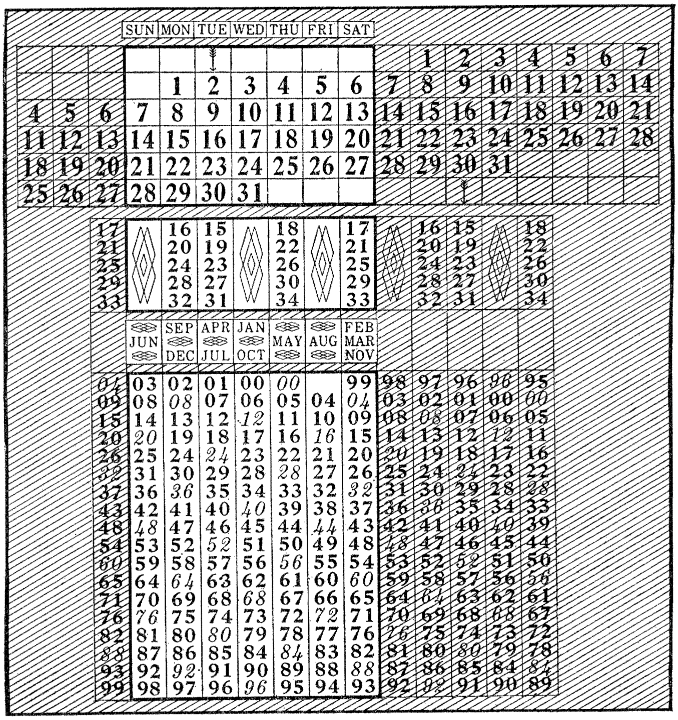
A short discussion of adjustable calendars is given by Irwin Roman[1] in the Monthly, *1915*, 241. He describes one of the several hundred mechanical devices used in presenting the calendar of a given month of a given year, but he does not give the mathematical theory upon which practically all perpetual calendars are based. It is the purpose of this article to present this theory.

There are approximately 365.2422 days in a tropical year. If each calendar year contained 365 days, in the course of 400 years an error of 96.88 days would accumulate. Then if 97 days be added for each period of 400 years the error will be very small. In any 400 consecutive integers there are 100 numbers which are divisible by 4, and 3 which are divisible by 100 but not by 400. Hence the rule: *Each year which is divisible by* 4 *but not by* 100 *and each year which is divisible by* 400 *is a leap year with* 366 *days. All other years contain* 365 *days.* Since this rule was first introduced by Pope Gregory XIII in the year 1582, no calendar based upon it need antedate the sixteenth century. The calendar may be extended into the future as far as one may choose. However, it becomes inaccurate after a few thousand years, because it accumulates a day in about 3,000 years, and also because the tropical year varies slightly as the years go by. Other calendars, which are used in some parts of the world or have been used in the past, are built upon other rules but I shall discuss only that calendar

[1] Mr. Roman's article begins with the statement, "So far as the writer has been able to learn, all perpetual or adjustable calendars are arranged so as to present the first day of the month as the first day of the week." He had evidently overlooked the most fruitful field for the study of perpetual calendars, viz., the *Official Gazette* of the United States Patent Office. During the past 25 years more than 100 calendars have been patented and more than a score of these present the days of the week in the natural order. For example see number 1048413, Dec. 24, 1912. [Doctor Morris's perpetual calendar was patented Nov. 26, 1918, number 1286058.—Editor.]

which results from the assumption of the above rule. The same general principles are used even though the rules are different.

Most perpetual calendars covering a period of several centuries contain five parts. These parts are *the days of the week*, *the days of the month*, *the months of the year*, *the years of the century* and *the centuries*. The days of the week are placed in regular order, beginning with any one of the seven days, and form an endless set of cycles of seven days each. The days of the month are the integers from 1 to 31 and are arranged in the usual manner. The months form an endless set of cycles of twelve months each. The years of the century, which occur in cycles of 100 years each, are indicated by the two right-hand digits of the numbers of the years; and the centuries are indicated by the numbers of the years after the years of the century have been displaced. They are the endless chain of integers which in our case begin with 15. The number which indicates the century is one less than its ordinal number. In the figure given herewith these five parts are shown from top to bottom in the order—days of the week, days of the month, centuries, months of the year, years of the century.



Let us start with the calendar for January, 1900, which is shown in the figure. Monday is the first day of the month. Since January has 31 days and $31 \equiv 3$ (mod. 7), *i.e.*, if 31 is divided by 7 the remainder is 3, February enters 3 days later in the cycle of days of the week than did January, which is Thursday. Hence the days of the week must be moved to the left 3 spaces or the days of the months to the right 3 spaces. Let us assume the latter movement and that the days of the week at all times remain in a fixed position. Also assume for the present that 00 is on a card which moves with the days of the month. It may be placed in any one of the seven columns. In the same column, but on the fixed card which contains the days of the week, place the month January. Now when the days of

the month are moved to the right 3 spaces 00 will be affected in the same manner. Then place February in the column containing the new position of 00, which is 3 spaces to the right of January. According to our rule February of 1900 contains 28 days. Thus the first day of March also falls on Thursday and March is in the same column as February. Since March has 31 days April is placed 3 columns to the right of March in the cycle of 7, which is one column to the left of January. This process may be continued until all the months of the year are located on the fixed card. When this is done the calendar for any month of the year 1900 is given when 00 is placed in the column of the desired month. The calendar shows 31 days for each month, but it is understood that only the proper number is to be used.

Now consider the year 1901. There are 365 days in 1900, and $365 \equiv 1 \pmod 7$. Hence every day of the year 1901 falls one day later in the week than the corresponding day of the year 1900, and 01 should be placed one space to the left of 00, causing the days of the months to be shifted one space to the right. When 01 is moved to the column of any month the calendar is given for that month in 1901. Likewise 02 is located one space to the left of 01, and 03 one to the left of 02. However, the extra day of 1904, which is a leap year, must be accounted for. January and February run regularly, but the remaining 10 months of the year enter *two* days later in the week than they did in 1903. Hence we must have two symbols for a leap year. Let the italic figures *04* placed one space to the left of 03, which is in the cycle 3 spaces to the right of 00, be used for January and February, and let the usual figures 04 be placed two spaces to the left of 03 and used for the remaining 10 months of the year. Similarly the rest of the years of the century can be located.

We still have to consider the shift which must be made in passing from one century to the next. There are 36525 days from March 1, 1900 to March 1, 2000; and $36525 \equiv 6 \pmod 7$. With the exception of the first two months, the months of the 21st century begin six days later in the week than did the corresponding months of the corresponding years of the 20th century. Hence in changing from 1999 to 2000 the days of the month must be moved 6 spaces to the right with respect to the years of the century. This can be accomplished if the days of the month are on an auxiliary slide which is mounted upon the main slide containing the years of the century. Now place 19 in any one of the columns on the main slide and 20 in the column 6 spaces to the right of 19. Before the auxiliary slide is moved from its original position place an arrow on it in the column containing 19. Then when the auxiliary slide is moved so that the arrow points to 20 the days of the month and the years of the century are in the correct relative position for all years of the 21st century. The first two months of the century are cared for by placing *00* in italics one space to the right of 00. There are 36524 days from March 1, 2000, to March 1, 2100; and $36524 \equiv 5 \pmod 7$. Hence the days of the month must be shifted 5 spaces to the right relative to the years of the century, which means that 21 should be 5 spaces to the right of 20 on the main slide. Likewise 22 should be 5 spaces to the right of 21, and 23

five to the right of 22; but 24 should be 6 spaces to the right of 23. The shift is 6 spaces for all centuries divisible by 4 and 5 spaces for the other centuries. This may be continued to any desired extent and may be extended backwards to the beginning of the calendar.

It is easy to see from the figure that there must be at least 13 columns on the main slide of this particular form of the calendar. With the given position of the centuries it is necessary to have 18 columns on the auxiliary slide. This number might have been reduced by one; but there would then have been a loss in symmetry in two parts. Other forms of the calendar allow different parts to move. A common type has the parts on disks, which rotate with respect to one another. Still another type uses one or more of the parts as reference tables.[1] However, the theory which I have given is applicable to practically all types.

II. Note on the Computation of Logarithms.

By Albert A. Bennett, University of Texas.

The following approximate relations are easily verified by the use of a computing machine or by direct multiplication with paper and pencil, and without recourse to logarithms. (The writer checked these on a "Monroe" in about two hours.)

(1) $(1.024)^{214} = 160.02580 \cdots,$

(2) $3\,(9/8)^{18} = 24.99577 \cdots,$

(3) $2 \times 5 = 10,$

(4) $7\,(98/81)^4 = 14.998996 \cdots,$

(5) $((2.2)^3\,(2.1))^2 = 500.005377 \cdots,$

(6) $39\,(13/14)^6 = 25.000945 \cdots,$

(7) $25\,(17/14)^5 = 66.000071 \cdots,$

(8) $20\,(190/33)^2 = 662.9936 \cdots,$

(9) $13\,(11/10)^2\,(17/13)^4 = 45.9992752 \cdots,$

(10) $551\,(55/221)^2\,(23/39)^3 = 6.9997431 \cdots.$

If these relations be replaced by the approximate relations

(1′) $(1.024)^{214} = 160,$

(2′) $3\,(9/8)^{18} = 25,$

(3′) $2 \times 5 = 10,$

(4′) $7\,(98/81)^4 = 15,$

(5′) $((2.2)^3\,(2.1))^2 = 500,$

(6′) $39\,(13/14)^6 = 25,$

(7′) $25\,(17/14)^5 = 66,$

(8′) $20\,(190/33)^2 = 663,$

(9′) $13\,(11/10)^2\,(17/13)^4 = 46,$

(10′) $551\,(55/221)^2\,(23/39)^3 = 7,$

[1] A very interesting example of this type is given by Augustus De Morgan in a volume of 88 pages entitled *The Book of Almanacs*. In addition to what is given by the ordinary perpetual calendar, his book includes saints' days, lunar calendars, several special calendars and discussions of historical interest.